

# **SANDIA REPORT**

SAND2009-6783P  
Unlimited Release  
Printed October 2009

## **Well-Posed Eigenvalue and Boundary Value Problems for the “Split” Turbulent Velocity Fluctuation Equations of L. J. Dechant, and Numerical Methods for their Solution**

Irina Kalashnikova, Sandia National Laboratories and Stanford University

Prepared by  
Sandia National Laboratories  
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under Contract DE-AC04-94-AL85000.

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Date: 10 October, 2009

To: Distribution

From: Irina Kalashnikova<sup>1,2</sup>

Subject: Well-Posed Eigenvalue and Boundary Value Problems for the  
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Numerical Methods for their Solution

<sup>1</sup>Aerosciences Department, Sandia National Laboratories, Albuquerque, NM.

<sup>2</sup>Institute for Computational & Mathematical Engineering (iCME), Stanford University, Stanford, CA.

*E-mail addresses:* ikalash@sandia.gov, irinak@stanford.edu (I. Kalashnikova)

### Abstract

The present document summarizes some recent (June – September 2009) extensions of a model for predicting transitional and fully turbulent pressure fluctuation loading using CFD mean flow information, proposed by L. J. De Chant in [4, 5]. The purpose of this work is to fine tune the equations at the heart of the model, and to formulate the most appropriate numerical method for their solution. Some shortcomings of the eigenvalue problem (EVP) proposed earlier [4, 5] are identified and addressed. Previously unseen connections to Sturm-Liouville and Orr-Sommerfeld theory are made. Several new eigenvalue and boundary value problems for the fluctuating quantities are proposed and evaluated in light of well-posedness, the underlying physics, and numerical implementation. The details of some suggested numerical solution methods (the series solution method and a Laguerre-Galerkin spectral method) are outlined. Preliminary numerical results for two EVPs formulated herein are presented. These results uncover some issues that remain and may need to be addressed in the future.

### DOE Funding Statement

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## 1 Introduction

Following the approach of [4, 5], we consider the 2D steady incompressible Euler equations. Introducing the classical Reynolds decomposition:

$$u = U + u', \quad v = V + v', \quad p = P + p' \quad (1)$$

where the capitalized quantities denote mean (temporal average) quantities and the ‘prime’ quantities denote temporal fluctuations, and substituting (1) into the equations, we obtain

$$[U_x + V_y] + [u'_x + v'_y] = 0 \quad (2)$$

$$[UU_x + VU_y + P_x + (u'u')_x + (u'v')_y] + [Uu'_x + Vu'_y + U_xu' + U_yv' + p'_x] = 0 \quad (3)$$

$$UV_x + VV_y + P_y + (u'v')_x + (v'v')_y + [Uv'_x + Vv'_y + V_xu' + V_yv' + p'_y] = 0 \quad (4)$$

(2) is the continuity relation and (3) and (4) are the  $x$ - and  $y$ -momentum expressions respectively.

As in [4, 5], the next step in the so-called “equation-splitting” approach is to “split” the mean equations from the fluctuating equations in (2)–(4). Doing so yields the following systems of equations:

$$\text{Mean: } \begin{cases} U_x + V_y = 0 \\ UU_x + VU_y + P_x + (u'u')_x + (u'v')_y = 0 \\ UV_x + VV_y + P_y + (u'v')_x + (v'v')_y = 0 \end{cases} \quad (5)$$

$$\text{Fluctuation: } \begin{cases} u'_x + v'_y = 0 \\ Uu'_x + Vu'_y + U_xu' + U_yv' + p'_x = 0 \\ Uv'_x + Vv'_y + V_xu' + V_yv' + p'_y = 0 \end{cases} \quad (6)$$

(5) will be referred to as the “mean equations” and (6) as the “fluctuating equations”. Note that we are *not* applying the averaging operator to (6) as is typically done in Reynolds-Averaged Navier-Stokes (RANS). Indeed, if we did this, then the equations (6) would be lost (i.e., the left-hand-sides would vanish).

If we make the “standard” boundary simplifications as done in [4, 5], namely assume  $(u'u')_x \approx (v'v')_y \approx (u'v')_x \approx VV_y \approx 0$ , and apply the eddy-viscosity model  $-(u'v')_y = \nu_{eff}U_{yy}$ , then the mean equations (5) become:

$$\text{Mean: } \begin{cases} U_x + V_y = 0 \\ UU_x + VU_y + P_x - \nu_{eff}U_{yy} = 0 \\ UV_x + VV_y + P_y = 0 \end{cases} \quad (7)$$

Note that with the boundary layer/modeling assumptions above, (7) has decoupled from the fluctuating equations (6). Thus, (7) can be solved either numerically or perhaps analytically [4, 5] for the mean quantities independently of (6).

The running document [7] contains several approximate analytic solutions to (7). One model for the mean flow velocities is the following exponential one:

$$U(x, y) = e^{-c_0x}(1 - e^{-b_0y}) \quad (8)$$

for some constants  $b_0, c_0$  (to be specified). With  $U(x, y)$  given by (8), one can approximate, when  $y \ll 1$ :

$$\frac{U_{xy}}{U} = -\frac{c_0b_0e^{-b_0y}}{1 - e^{-b_0y}} \approx -\frac{c_0}{y}, \quad y \ll 1 \quad (9)$$

$$\frac{U_{yy}}{U} = -\frac{b_0^2e^{-b_0y}}{1 - e^{-b_0y}} \approx -\frac{b_0}{y}, \quad y \ll 1 \quad (10)$$

The approximations (9) and (10) will be employed in Sections 3.2 and 4.2 for the sake of obtaining some preliminary numerical results.

Given the equation splitting (5) and (6) our approach will be to solve these equations by decoupling the mean equations as in (7), solving for the mean quantities, substituting them into (6) and solving for the fluctuations from (6). Of course, as these equations stand, they are incomplete: boundary conditions are required. We will take as our domain the following semi-infinite region:

$$\Omega \equiv (0, \infty) \times (0, \infty) \subset \mathbb{R}^2 \quad (11)$$

The boundary conditions on the mean velocities are

$$U(x, 0) = V(x, 0) = V(x, \infty) = 0, \quad U(x, \infty) = 1 \quad (12)$$

The boundary conditions on the fluctuations are

$$u'(x, 0) = u'(x, \infty) = v'(x, 0) = v'(x, \infty) = p'(x, 0) = p'(x, \infty) = 0 \quad (13)$$

In two-dimensions (2D), (6) and (13) can be written as a third order, scalar equation for the fluctuating streamfunction  $\psi'$ , defined by

$$u' = \psi'_y, \quad v' = -\psi'_x \quad (14)$$

The streamfunction BVP equivalent to (6) with boundary conditions (13) is:

$$\begin{cases} U\psi'_{xxx} + V\psi'_{yyy} + U\psi'_{xyy} + V\psi'_{xxy} + (U_{xy} - V_{xx})\psi'_y + (V_{xy} - U_{yy})\psi'_x &= 0 \\ \psi'_x(x, 0) = \psi'_x(x, \infty) = \psi'_y(x, 0) = \psi'_y(x, \infty) &= 0 \end{cases} \quad (15)$$

Once (15) is solved for the fluctuating streamfunction  $\psi'$ , one can obtain the fluctuating velocities by differentiating this function per (14), and then solve the following pressure-Poisson equation for the pressure fluctuation:

$$\begin{cases} p'_{xx} + p'_{yy} &= -2(V_x u'_y + U_x u'_x + V_y v'_y + U_y v'_x) \\ p'(x, 0) &= 0 \\ p'(x, \infty) &= 0 \end{cases} \quad (16)$$

At this point, the following observation is in order: (6) with the boundary conditions (13) (or, equivalently, the streamfunction BVP (15)) is ill-posed, as it is a homogeneous steady partial differential equation (PDE) with homogeneous boundary conditions. In particular, if (6) with (13) (or equivalently (15)) were to be solved numerically, one would obtain the trivial, uninteresting solution  $u' = v' = p' = 0$ .

The aim of this document is to propose various ways of remedying this problem of ill-posedness of the fluctuating equations. We propose several ways to do this:

- Given the mean flow, formulate (15) as a Sturm-Liouville eigenvalue problem (EVP) by assuming an ansatz (solution form)  $\psi'(x, y) = \hat{\psi}(y)e^{\alpha x}$ , with  $\alpha$  playing the role of an (unknown) eigenvalue to be solved for, much like in classical Orr-Sommerfeld theory (Section 3).
- Given the mean flow, add fluctuating time scale terms  $u'_t$  and  $v'_t$  to (6) to yield an *initial* boundary value problem (IBVP)

$$\begin{cases} u'_x + v'_y &= 0 \\ u'_t + Uu'_x + Vu'_y + U_x u' + U_y v' + p'_x &= 0 \\ v'_t + Uv'_x + Vv'_y + V_x u' + V_y v' + p'_y &= 0 \end{cases} \quad (17)$$

with some non-trivial initial conditions  $u'(0, x, y) = u'_0(x, y)$ ,  $v'(0, x, y) = v'_0(x, y)$ ,  $p'(0, x, y) = p'_0(x, y)$  (Section 4). The IBVP (17) can be formulated as an eigenvalue problem by assuming an unsteady Orr-Sommerfeld-like ansatz  $\psi'(t, x, y) = \hat{\psi}(y)e^{\lambda t + \alpha x}$  for the streamfunction (Section 4).

- Introduce a mean-flow-dependent source into the homogeneous fluctuating equations (6) (Sections 5).

Although simplified variants of the equations can be solved analytically under some assumptions [4, 5, 7], ultimately the fluctuations will be solved for numerically. The numerical method to be employed depends on what equations are ultimately selected, as different formulations of the equations are amenable to different solution methods. As we show below, the steady Sturm-Liouville eigenvalue problem (point 1 above) suggests a series solution approach (Section 3). A formulation involving unsteady fluctuations (point 2 above) can be expressed as an eigenvalue problem that is amenable to solution by a Laguerre-Galerkin spectral method (Section 4).

As illuminated by preliminary numerical experiments (Sections 3.2 and 4.2), an important question that needs to be addressed, once a set of equations is selected and solved, is precisely how to interpret and validate the solutions to these equations. Some issues that arise include in particular:

- The possibility of complex solutions (eigenvalues and eigenfunctions), and what these solutions would mean physically.
- Whether it is appropriate to associate the fluctuating quantities that solve (6) with root-mean-square (rms) quantities.
- Whether *linear* equations for the fluctuations like (6) accurately describe the physics, turbulence being an inherently *nonlinear* phenomenon.

In what is to come, we present several well-posed variants of the “split” equations for the fluctuating quantities. For simplicity, and without loss of generality, in Sections 3 and 4, we focus on the streamfunction equation (15). The streamfunction approach *is* limited to two-dimensions (2D); however we emphasize that the *methods* presented herein

are not limited to 2D and the scalar BVP (15), i.e., they can just as easily be applied to the equations in the primitive variables (6), or the three-dimensional (3D) version of these equations. Care is taken to formulate the equations such that they are well-posed and amenable to numerical solution by an easy-to-implement and appropriate numerical method. The details of some of these implementations are laid out, and some preliminary numerical results for two of the proposed eigenvalue problems (EVPs) are given and interpreted. We conclude with a discussion of some issues that remain and may need to be addressed in the future.

## 2 The “Old” Eigenvalue Problem (EVP) [4, 5] and the Sturm-Liouville Connection

The basic idea behind the approach taken in subsequent sections (Sections 3-4) rests on the observation that (15) is ill-posed with the prescribed boundary conditions – *unless* it is viewed as an eigenvalue problem (EVP), like in standard Sturm-Liouville theory. Recall that, in general, the Sturm-Liouville problem is given by:

$$-[p(x)u'(x)]' + q(x)u(x) = \lambda w(x)u(x) \quad (18)$$

where  $u(x)$  is the unknown function (solution),  $p(x)$ ,  $q(x)$  and  $w(x)$  are prescribed, and  $\lambda$  is an (unknown scalar) eigenvalue. The solutions to (18) define various families of orthogonal polynomials, orthogonal in the  $L^2(\Omega)$  inner product with respect to the weight  $w(x)$ . For example, when  $p(x) = 1$ ,  $q(x) = 0$ ,  $w(x) = 1$ , the solutions to (18) define the Fourier basis; when  $p(x) = 1 - x^2$ ,  $q(x) = 0$ ,  $w(x) = 1$ , the solutions to (18) define the Legendre basis, etc.

Motivated by this Sturm-Liouville/eigenvalue problem connection, an attempt was made in [4, 5] to formulate (15) as an eigenvalue problem resembling (18). Denoting  $\varepsilon = V/U$ , dividing through by  $U$  and assuming  $V_{xx} = V_{xy} = 0$ , (15) becomes:

$$\begin{cases} \psi'_{xxx} + \varepsilon \psi'_{yyy} + \psi'_{xyy} + \varepsilon \psi'_{xxy} + \frac{U_{xy}}{U} \psi'_y - \frac{U_{yy}}{U} \psi'_x &= 0 \\ \psi'_x(x, 0) = \psi'_x(x, \infty) = \psi'_y(x, 0) = \psi'_y(x, \infty) &= 0 \end{cases} \quad (19)$$

In order to make (19) resemble (18) and ensure well-posedness of this BVP, an ad-hoc eigenvalue  $\lambda$  was introduced, based on scaling arguments. This was done by replacing

$$\left\{ \frac{U_{xy}}{U} \psi'_y - \frac{U_{yy}}{U} \psi'_x \right\} \leftarrow \{ \lambda U''(x, y) [\psi'_x + \psi'_y] \} \quad (20)$$

where  $U''(x, y)$  is a function describing the behavior of the mean velocity gradients, to be specified (modeled), and  $\lambda$  is an unknown eigenvalue parameter, to be solved for in solving (21). Making the substitution (20) in (19), the following EVP was obtained in [4, 5]:

$$\begin{cases} \psi'_{xxx} + \varepsilon \psi'_{yyy} + \psi'_{xyy} + \varepsilon \psi'_{xxy} + \lambda U''(x, y) [\psi'_y + \psi'_x] &= 0 \\ \psi'_x(x, 0) = \psi'_x(x, \infty) = \psi'_y(x, 0) = \psi'_y(x, \infty) &= 0 \end{cases} \quad (21)$$

We emphasize that the eigenvalue  $\lambda$  in (21) is added as an additional unknown degree of freedom to ensure well-posedness of (19).

Recent work has revealed that the “ad hoc” EVP (21) has some shortcomings. While the EVP itself is mathematically well-defined in the sense that it permits a non-trivial solution, since the *eigenvalue*  $\lambda$  in (21) was introduced in an ad hoc, artificial fashion (20), it is unclear what this eigenvalue and its corresponding eigenfunctions mean. In particular, it is unclear what to make of complex eigenfunctions and eigenvalues. Numerical implementation using a Hermite cubic finite element discretization (similar to the approach outlined in [3]) revealed that the eigenvalues and their corresponding eigenfunctions *were* in general complex, and indeed, as the operator governing (21) is asymmetric, real-valued solutions to (21) cannot be guaranteed.

### 3 Steady Eigenvalue Problem (EVP) for the Fluctuations with Orr-Sommerfeld-Like Ansatz

Given the difficulty in interpreting the solutions to the “old” eigenvalue problem (21), we seek a more natural EVP, in which the meaning of the eigenvalues and eigenfunctions are more clearly defined and tied to the equation and/or the physics in some way. To do so, it is useful to make a connection to Orr-Sommerfeld theory.

Recall that, for the well-known Orr-Sommerfeld equation (Section 5.2 of [12]), one obtains an eigenvalue problem by assuming an ansatz of the form

$$\begin{aligned} u'(t, x, y) &= \hat{u}(y)e^{i\alpha(x-ct)} \\ v'(t, x, y) &= \hat{v}(y)e^{i\alpha(x-ct)} \\ p'(t, x, y) &= \hat{p}(y)e^{i\alpha(x-ct)} \end{aligned} \quad (22)$$

and substituting (22) into (in the simplest case) the unsteady 2D Euler equations. Doing so yields the well-known Orr-Sommerfeld EVP:

$$\begin{aligned} \hat{v}'' - \left( \frac{U_{xy}}{U-c} + \alpha^2 \right) \hat{v} &= 0 \\ \hat{v}(0) &= \hat{v}(\infty) = 0 \end{aligned} \quad (23)$$

The three unknown parameters in (23) are  $\hat{v}$  (the primal unknown), and  $\alpha \in \mathbb{C}$  and  $c \in \mathbb{C}$  (a sequence of eigenvalues). These eigenvalues  $\alpha$  and  $c$  are related to stability: from (22), one can see that the sign of their imaginary parts determines whether the solution grows temporally and/or spatially. Once the unknowns in (23) are computed, the final solution can be obtained by substituting these functions and eigenvalues into (22). Hence, the eigenvalues and eigenfunctions have a straight forward connection to the solutions of the original equations.

One may apply the Orr-Sommerfeld ansatz approach outlined above to derive an eigenvalue problem (EVP) for the fluctuating streamfunction. Begin by assuming the streamfunction has the following functional form:

$$\psi'(x, y) = \hat{\psi}(y)e^{\alpha x} \quad (24)$$

Here,  $\alpha \in \mathbb{C}$  is a scalar that can be thought of as a wave number of the disturbance. Substituting the ansatz (24) into (19), gives

$$\boxed{\begin{aligned} \varepsilon \hat{\psi}''' + \alpha \hat{\psi}'' + \left( \varepsilon \alpha^2 + \frac{U_{xy}}{U} \right) \hat{\psi}' + \alpha \left( \alpha^2 - \frac{U_{yy}}{U} \right) \hat{\psi} &= 0 \\ \hat{\psi}(0) = \hat{\psi}(\infty) = \hat{\psi}'(0) = \hat{\psi}'(\infty) &= 0 \end{aligned}} \quad (25)$$

where  $\varepsilon \equiv V/U$ . The mean  $x$ -velocity and its gradients, namely  $U$ ,  $U_{xy}$  and  $U_{yy}$  that appear in (25) are to be fed in from the code that solves the (decoupled) mean equations (5), or modeled, as in, e.g., (9)–(10).

The “natural” EVP (25) and “artificial” EVP (21) are different, as expected. Rather than introducing an ad hoc scaling parameter to represent the eigenvalue, we have employed Orr-Sommerfeld analysis, with the parameter  $\alpha$  in (24) representing the eigenvalue. Once the solutions to (25) (the eigenvalues  $\alpha$  and corresponding eigenfunctions  $\hat{\psi}$ ) are obtained, the final solution is given by (24).

Note that (25) is nonlinear in the eigenvalue  $\alpha$ . As we will show below in Sections 3.1–3.2, one may derive a series solution to (25) that can be implemented easily, e.g., in MATLAB. A “direct” numerical solution of (25) by a standard discretization (e.g., using a spectral method; see Section 4.1) would require the application of Newton’s method to handle the non-linearities. It is interesting to observe that under the assumption that  $\varepsilon \approx 0$  and  $U = U(y)$  only, (25) simplifies to:

$$\hat{\psi}'' - \frac{U_{yy}}{U} \hat{\psi} + \alpha^2 \hat{\psi} = 0 \quad (26)$$

(26) is a linear EVP very similar to (23) that can be solved numerically (or perhaps analytically for simple enough choices of  $U$ ) with ease using a spectral method like the Laguerre-Galerkin method outlined in Section 4.1.

### 3.1 Series Solution Approach to the EVP (25)

Given the connection to Sturm-Liouville exhibited above, it is natural to try classical Sturm-Liouville solution techniques to try to derive analytically solutions to (25). One standard technique is the series expansion partial differential equation (PDE) solution technique. This is, in fact, one way of deriving the nice families of orthogonal polynomials that solve (18), e.g., the Fourier, Legendre, Hermite, Laguerre, etc. bases. The approach is as follows: begin by assuming a solution of the form

$$\hat{\psi}(y) = \sum_{m=1}^{\infty} a_m y^m e^{-\beta y} \quad (27)$$

Then substitute the series (27) into the EVP (25), and derive a recursive relation for the coefficients  $a_m$ . The weight<sup>1</sup>  $e^{-y}$  has been added to (27) so that it is possible to satisfy the homogeneous boundary condition on  $u'$  and  $v'$  at  $y = \infty$ .

For the purpose of generating some actual analytical and numerical results, assume that, for  $y \ll 1$ :

$$\frac{U_{xy}}{U} = -\frac{c_0}{y}, \quad \frac{U_{yy}}{U} = -\frac{b_0}{y} \quad (28)$$

for some specified (modeled) constants  $c_0, b_0 \in \mathbb{R}$  (see the running document [7] and (9)–(10) above). With the mean velocities given by (28), (25) becomes (multiplying through by  $y$  to avoid singular coefficients):

$$\boxed{\varepsilon y \hat{\psi}''' + \alpha y \hat{\psi}'' + (\varepsilon \alpha^2 y - c_0) \hat{\psi}' + \alpha (\alpha^2 y + b_0) \hat{\psi} = 0} \quad (29)$$

Differentiating (27), we obtain:

$$\hat{\psi}'(y) = -\beta \sum_{m=1}^{\infty} a_m y^m e^{-\beta y} + \sum_{m=2}^{\infty} m a_m y^{m-1} e^{-\beta y} \quad (30)$$

$$\hat{\psi}''(y) = \beta^2 \sum_{m=1}^{\infty} a_m y^m e^{-\beta y} - 2\beta \sum_{m=2}^{\infty} m a_m y^{m-1} e^{-\beta y} + \sum_{m=3}^{\infty} m(m-1) a_m y^{m-2} e^{-\beta y} \quad (31)$$

$$\begin{aligned} \hat{\psi}'''(y) &= -\beta^3 \sum_{m=1}^{\infty} a_m y^m e^{-\beta y} + 3\beta^2 \sum_{m=2}^{\infty} m a_m y^{m-1} e^{-\beta y} - 3\beta \sum_{m=3}^{\infty} m(m-1) a_m y^{m-2} e^{-\beta y} \\ &\quad + \sum_{m=4}^{\infty} m(m-1)(m-2) a_m y^{m-3} e^{-\beta y} \end{aligned} \quad (32)$$

Substituting (27) and (30)–(32) into (29) and re-indexing:

$$\begin{aligned} & -\sum_{m=2}^{\infty} \beta^3 \varepsilon a_{m-1} y^m + \sum_{m=2}^{\infty} 3\beta^2 \varepsilon m a_m y^m - \sum_{m=2}^{\infty} 3\beta \varepsilon (m+1) m a_{m+1} y^m \\ & + \sum_{m=2}^{\infty} \varepsilon (m+2)(m+1) m a_{m+2} y^m + \sum_{m=2}^{\infty} \beta^2 \alpha a_{m-1} y^m - \sum_{m=2}^{\infty} 2\beta \alpha m a_m y^m \\ & + \sum_{m=2}^{\infty} \alpha (m+1) m a_{m+1} y^m - \sum_{m=2}^{\infty} \varepsilon \beta \alpha^2 a_{m-1} y^m + \sum_{m=2}^{\infty} \varepsilon \alpha^2 m a_m y^m \\ & + \sum_{m=1}^{\infty} c_0 \beta a_m y^m - \sum_{m=1}^{\infty} c_0 (m+1) a_{m+1} y^m + \sum_{m=2}^{\infty} \alpha^3 a_{m-1} y^m + \sum_{m=1}^{\infty} \alpha b_0 a_m y^m = 0 \end{aligned} \quad (33)$$

or

$$\begin{aligned} & [(\beta c_0 + \alpha b_0) a_1 - 2c_0 a_2] y e^{-\beta y} + \sum_{m=2}^{\infty} \{ [-\beta^3 \varepsilon + \beta^2 \alpha - \beta \varepsilon \alpha^2 + \alpha^3] a_{m-1} \\ & + [3\beta^2 \varepsilon m - 2\beta \alpha m + \varepsilon \alpha^2 m + \beta c_0 + \alpha b_0] a_m + [-3\beta \varepsilon (m+1) m + \alpha (m+1) m - c_0 (m+1)] a_{m+1} \\ & + \varepsilon (m+2)(m+1) m a_{m+2} \} y^m e^{-\beta y} = 0 \end{aligned} \quad (34)$$

(34) holds if each of the coefficients in the sum is zero for each  $m$ , i.e., if the  $a_m$  satisfy the following recursion relations:

$$(\beta c_0 + \alpha b_0) a_1 - 2c_0 a_2 = 0 \quad (35)$$

$$\begin{aligned} & [-\beta^3 \varepsilon + \alpha^3 + \beta \alpha (\beta - \varepsilon \alpha)] a_{m-1} + [\beta (3\beta \varepsilon m - \alpha m + c_0) - \alpha (\beta - \varepsilon \alpha) m + \alpha b_0] a_m \\ & + [-3\beta \varepsilon m + \alpha m - c_0] (m+1) a_{m+1} + \varepsilon (m+2)(m+1) m a_{m+2} = 0, \quad m = 2, 3, 4, \dots \end{aligned} \quad (36)$$

with  $\psi'(x, y)$  given by (from (27) and (24))

$$\psi'(x, y) = \sum_{m=1}^{\infty} a_m y^m e^{-\beta y} e^{\alpha x} \quad (37)$$

<sup>1</sup>Note that (27) is somewhat reminiscent of the series solution assumed for Laguerre's differential equation, which generates the so-called Laguerre functions; see Section 4.1 for more on the Laguerre spectral basis.

(36) defines a matrix with four nonzero diagonals, which reduces to a tridiagonal system when  $\alpha = \varepsilon$ . If additional assumptions are made to simplify (25) it may be possible to derive a nice recursive relation for the  $a_m$  by simplifying (36).

Note that (37), although based on the idea of an EVP with an unknown eigenvalue, does not require solution for  $\alpha$  or  $\beta$ , or any eigenvalue for that matter.  $\alpha$  and  $\beta$  are to be modeled (prescribed). In the implementation, one can simply code the solution  $\psi'(x, y)$  as in (37) with the  $a_m$  given by (36). The coefficient  $\beta$  would be specified to ensure that the far-field homogeneous boundary conditions are satisfied. Noting that

$$e^{-\beta y} = 1 - \beta y + \frac{(\beta y)^2}{2!} - \frac{(\beta y)^3}{3!} + \dots \quad (38)$$

one would ensure satisfaction of the far-field condition if  $\beta$  is set such that:

$$\beta > \max_{m=1, \dots, M} (|a_m| m!)^{1/m} \quad (39)$$

One question that may arise is how to handle the case when the expressions (models) for  $U_{yy}/U$  and  $U_{xy}/U$  are more complicated than (28), or if they are functions of  $x$ . A natural remedy here would be to freeze  $x$ , write a Taylor series for these quantities, and apply the series approach using these Taylor-expanded quantities for each fixed (frozen) value of  $x$ .

### 3.2 Some Numerical Results to (25) using the Series Approach

Figures 1–2 show plots of the partial sum solutions

$$\psi'_M(x, y) \equiv \sum_{m=0}^M a_m y^m e^{-\beta y} e^{\alpha x} \quad (40)$$

with

$$\varepsilon = 0.01, \quad \alpha = -\varepsilon, \quad \beta = 1, \quad c_0 = 0.001, \quad b_0 = 1, \quad a_1 = 0.1 \quad (41)$$

plotted for  $x, y \in (0, 50) \times (0, 50)$  with  $M = 50$  in the partial sum (40). Note that the constant  $a_1$  in (35) is what specifies the magnitude of the fluctuating velocity profiles (the “normalization constant”). A question that needs to be addressed is how to normalize the solutions, i.e., how to fix the constant  $a_1$  specifying the magnitude of the velocity fluctuations.

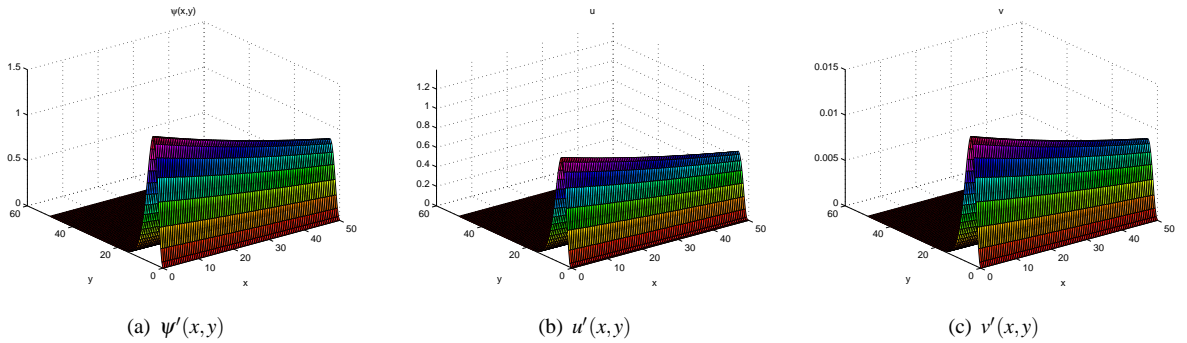


Figure 1: Fluctuating quantities (40) with (41),  $M = 50$  plotted on  $x, y \in (0, 50) \times (0, 50)$

Convergence of the series (37) is of interest, i.e., the convergence of the coefficients  $a_m$ . Figure 3 is a preliminary numerical validation that shows that for the values of the properties (41), the  $a_m \rightarrow 0$  as  $m \rightarrow \infty$ . If the series approach is adopted in practice, it would be worthwhile to analyze the convergence of (40) as  $M \rightarrow \infty$  (the decay of the coefficients  $a_m$ ), analytically well as numerically.



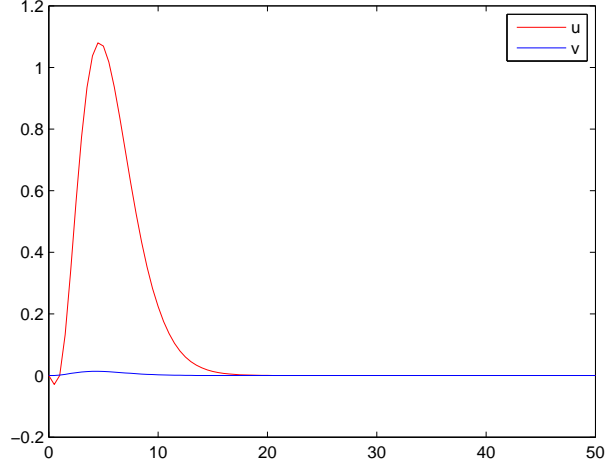


Figure 2:  $u'$  and  $v'$  (40) with (41),  $M = 50$  plotted on  $y \in (0, 50)$  along the cross-section  $x = 0.5$

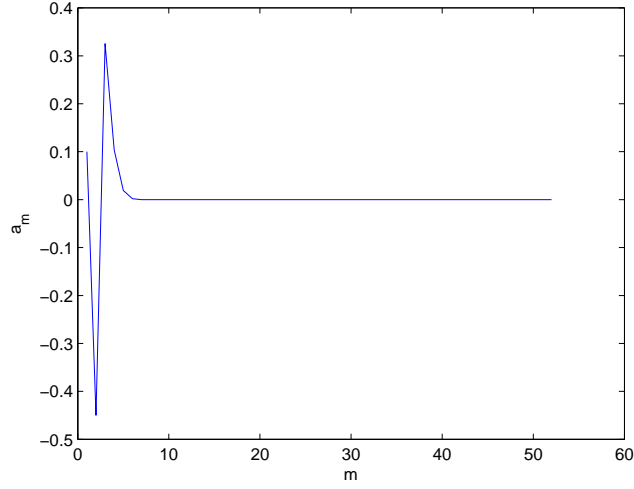


Figure 3: Decay of the coefficients  $a_m$  ( $M = 50$ )

As a final comment, as the plots above (Figures 1–2) suggest,  $u'$  and  $v'$  are actually *not* root-mean-square (rms) quantities, a connection that was made in [4, 5] for purposes of model validation. Indeed, as the original equations (6) was not derived for rms quantities, there is no reason to associate the solutions to this EVP with rms velocities or require them to be non-negative. An appropriate interpretation of the solutions to (6) is currently lacking. Are these solutions simply pointwise velocity fluctuations? Some kind of physical interpretation will be required both for validation of the model/governing equations selected, and for ultimate calculation of the desired turbulent pressure loads.

## 4 “Pseudotemporal” Eigenvalue Problem (EVP) for the Fluctuations with Orr-Sommerfeld-Like Ansatz

There is another way to handle the ill-posedness of the fluctuating equations (6). As suggested in the introduction, one can assume the fluctuating quantities are unsteady:  $u'(t, x, y), v'(t, x, y), p'(t, x, y)$ . In this case, the fluctuating equations are actually (17), that is, they are (6) but with a time derivative appearing in the two momentum equations. The unsteady BVP (17), although homogeneous, is *not* ill-posed, provided one specifies non-trivial initial conditions  $u'(0, x, y) = u'_0(x, y), v'(0, x, y) = v'_0(x, y), p'(0, x, y) = p'_0(x, y)$ . Rewriting (17) in terms of the fluctuating streamfunction  $\psi'(t, x, y)$  (14), one obtains in place of (19) the following unsteady PDE:

$$(\psi'_{yy} + \psi'_{xx})_t + V\psi'_{yyy} + U\psi'_{xxx} + U\psi'_{xyy} + V\psi'_{xxy} + U_{xy}\psi'_y - U_{yy}\psi'_x = 0 \quad (42)$$

In what we will refer to as the “pseudotemporal” approach, we use the unsteady terms in (42) to generate a well-posed EVP using Orr-Sommerfeld theory. We call the approach *pseudotemporal* because in actuality, we are interested in the steady fluctuations. Thus, time is added to (42) *for the sake of well-posedness only*, to be taken away once a non-trivial solution is obtained, following the procedure outlined in the running document [7], as well as below.

Given (42), the “pseudotemporal” approach says to introduce a time-dependent ansatz for the fluctuating streamfunction of the form

$$\psi'(t, x, y) = \hat{\psi}(y)e^{\alpha x}e^{\lambda t} \quad (43)$$

Substituting (43) into (42), we obtain:

$$\boxed{\begin{aligned} \varepsilon U \hat{\psi}''' + \alpha U \hat{\psi}'' + (\varepsilon \alpha^2 U + U_{xy}) \hat{\psi}' + \alpha(\alpha^2 U - U_{yy}) \hat{\psi} + \lambda(\hat{\psi}'' + \alpha^2 \hat{\psi}) &= 0 \\ \hat{\psi}(0) = \hat{\psi}(\infty) = \hat{\psi}'(0) = \hat{\psi}'(\infty) &= 0 \end{aligned}} \quad (44)$$

where  $\varepsilon \equiv V/U$ . (44) can be viewed, and solved numerically, as an eigenvalue problem, for the unknown eigenvalues  $\lambda_i$  and their corresponding eigenfunctions  $\hat{\psi}_i$ . Unlike the EVP (25), the EVP (45) is *linear* in the eigenvalue, as in (45) the eigenvalue comes from the “pseudo”-time dependence, not from the assumed behavior in the  $x$ -direction. This makes (44) quite amenable to numerical solution using a spectral method with basis functions defined on a semi-infinite domain, e.g., a Laguerre-Galerkin method. The details of the numerical solution of (44) by a Laguerre-Galerkin spectral method is outlined in Section 4.1, where we also give some numerical results for a simple choice of the parameters.

In the “pseudotemporal” approach, once the eigenvalues and eigenfunctions are computed, the solution (in actuality a steady function) is set to be:

$$\psi'(x, y) = \text{span}\{\hat{\psi}_i(y)\}e^{\alpha x} \quad (45)$$

(45) amounts to essentially setting  $t \equiv \text{const}$  in (43). The parameter  $\alpha$  in (43) is to be specified *a priori* (modeled), as are  $U$ ,  $U_{yy}$  and  $U_{xy}$ .

We end by calling to the reader’s attention the fact that what is essentially done in the “pseudotemporal” approach is a source is implicitly introduced into the homogeneous steady equations (19). It is easiest to demonstrate this on a simpler PDE. Suppose we wish to apply the “pseudotemporal” approach to the one-dimensional (1D) heat equation on  $x \in (0, 1)$ :

$$\begin{aligned} u_{xx} &= 0 \\ u(0) = u(1) &= 0 \end{aligned} \quad (46)$$

Of course, the solution to (46) is trivial. The “pseudotemporal” approach says to add a derivative with respect to time

$$\begin{aligned} u_t + u_{xx} &= 0 \\ u(0) = u(1) &= 0 \end{aligned} \quad (47)$$

and assume an ansatz of the form  $u(x, t) = \hat{u}(x)e^{\lambda t}$ . Substituting this ansatz into (47) gives the EVP:

$$\begin{aligned} \hat{u}'' + \lambda \hat{u} &= 0 \\ \hat{u}(0) = \hat{u}(1) &= 0 \end{aligned} \quad (48)$$

The solution to (48) is the span of Fourier sine functions:  $\hat{u}(x) = \text{span}\{\sin(n\pi x) : n \in \mathbb{Z}\}$ . Substituting this into the ansatz and setting  $t = 0$ , following the “pseudotemporal” approach, we get as our non-trivial “pseudotemporally” computed solution:

$$u(x) = \text{span}\{\sin(n\pi x) : n \in \mathbb{Z}\} \quad (49)$$

Note that (49) does *not* solve the original PDE (46); rather it solves

$$\begin{aligned} u_{xx} &= \lambda_n^2 \sin(\lambda_n x) \\ u(0) &= u(1) = 0 \end{aligned} \quad (50)$$

where  $\lambda_n = n\pi$  for  $n \in \mathbb{Z}$ . Thus, the “pseudotemporal” approach ensures well-posedness by adding “artificial” unsteady terms that essentially generate an artificial source for the original homogeneous PDE. This idea is quite novel; therefore its appropriateness and validity in the context of the physics inherent in our model is worth investigating further in future research. In addition, it motivates Section 5, where a source is introduced directly into the fluctuating equations (6).

#### 4.1 Laguerre-Galerkin Spectral Method Solution to the EVP (44)

Since (44) (and also (25), but here we focus our attention on (44)) is posed on a semi-infinite domain, it is amenable to numerical solution by a Laguerre-Galerkin spectral method.

Recall the Laguerre polynomials, defined by:

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1 - x \\ (n+1)L_{n+1}(x) &= (2n+1-x)L_n(x) - nL_{n-1}(x), \quad n = 1, 2, \dots \end{aligned} \quad (51)$$

These functions satisfy Laguerre’s equation,  $xL_n'' + (1-x)L_n' + nL_n = 0$ , with the boundary condition  $L_n(0) = 1$ . They are orthogonal in the  $L^2([0, \infty))$  inner product with respect to the weight  $e^{-x}$ .

To solve (44) by a Galerkin method, we are required, by definition, to have a basis that satisfies the boundary conditions, in this case, a homogeneous BC on  $\hat{\psi}$  and  $\hat{\psi}'$  at 0 and  $\infty$ . It is clear that the Laguerre polynomials (51) do not satisfy these boundary conditions. It takes some work to come up with appropriate basis functions that satisfy the required boundary condition (see as a reference [10]). It turns out that the following basis  $\{\hat{\phi}_i\}_{i=0}^N$  does the trick:

$$\boxed{\hat{\phi}_i(x) \equiv [L_i(x) - L_{i+1}(x) + 2x]e^{-x/2}} \quad (52)$$

It is straight forward to check that  $\hat{\phi}_i(0) = \hat{\phi}_i(\infty) = \hat{\phi}_i'(0) = \hat{\phi}_i'(\infty) = 0$  for all  $i = 0, 1, 2, \dots$ . We will therefore expand our streamfunction  $\hat{\psi}(y)$  in the basis (52), so that we will solve for the coefficients  $a_n$  such that:

$$\boxed{\hat{\psi}(y) \approx \hat{\psi}_N(y) \equiv \sum_{n=0}^{N-1} a_n \hat{\phi}_n(y)} \quad (53)$$

with the  $\hat{\phi}_n$  defined in (52).

Projecting (44) onto the  $i^{th}$  basis function  $\hat{\phi}_i$  and performing some integrations by parts, we obtain the following weak formulation of the EVP:

$$\begin{aligned} \int_0^\infty (U_y \hat{\phi}_i + U \hat{\phi}_i')(\varepsilon \hat{\psi}'' + \alpha \hat{\psi}') dy - \int_0^\infty (\varepsilon \alpha^2 U + U_{xy}) \hat{\psi}' \hat{\phi}_i dy - \alpha \int_0^\infty (\alpha^2 U - U_{yy}) \hat{\psi} \hat{\phi}_i dy \\ = \lambda [\alpha^2 \int_0^\infty \hat{\psi} \hat{\phi}_i dy - \int_0^\infty \hat{\psi}' \hat{\phi}_i' dy] \end{aligned} \quad (54)$$

In matrix form, denoting the vector of unknowns by  $\mathbf{a}^T \equiv (a_0 \quad \dots \quad a_{N-1}) \in \mathbb{R}^N$ , (54) can be written as

$$\mathbf{Ka} = \lambda \mathbf{Ma} \quad (55)$$

with the stiffness and mass matrix entries defined respectively by:

$$K(i, j) \equiv \int_0^\infty (U_y \hat{\phi}_i + U \hat{\phi}_i') (\varepsilon \hat{\phi}_j'' + \alpha \hat{\phi}_j') dy - \int_0^\infty (\varepsilon \alpha^2 U + U_{xy}) \hat{\phi}_j' \hat{\phi}_i dy - \alpha \int_0^\infty (\alpha^2 U - U_{yy}) \hat{\phi}_j \hat{\phi}_i dy \quad (56)$$

$$M(i, j) \equiv \alpha^2 \int_0^\infty \hat{\phi}_j \hat{\phi}_i dy - \int_0^\infty \hat{\phi}_j' \hat{\phi}_i' dy \quad (57)$$

The general solution procedure is as follows:

1. Specify  $\alpha$ ,  $\varepsilon$ ,  $U$ ,  $U_{xy}$  and  $U_{yy}$ .
2. Select  $N$ , the number of basis functions you wish to use.
3. Compute the stiffness and mass matrices from (56) and (57) given the basis functions (52).
4. Solve the generalized discrete EVP (55), e.g., in MATLAB, with the command `[A, L] = eig(K, M)`.
5. Set as the steady, pseudo-temporally computed streamfunction solution:

$$\psi_N'(x, y) = \text{span}_{j=1, \dots, N} \left\{ \sum_{i=1}^N A(i, j) \hat{\phi}_i(y) \right\} e^{\alpha x} \quad (58)$$

## 4.2 Some Numerical Results to (44) using the Laguerre-Galerkin Spectral Method

As a numerical experiment, to get some idea of what the solutions to (44) may look like, let us fix:

$$\varepsilon = 0.01, \quad \alpha = -\varepsilon, \quad \frac{U_{xy}}{U} = -\frac{c_0}{y}, \quad \frac{U_{yy}}{U} = -\frac{b_0}{y}, \quad c_0 = 0.001, \quad b_0 = 1 \quad (59)$$

The results are quite interesting. First, observe that the eigenvalues seem to decay to zero (Figure 4). Although most of the eigenvalues have a non-zero imaginary part, the magnitude of the imaginary part is in general much smaller than the magnitude of the real part. This may be grounds for omitting the imaginary parts of the eigenfunctions in practical applications.

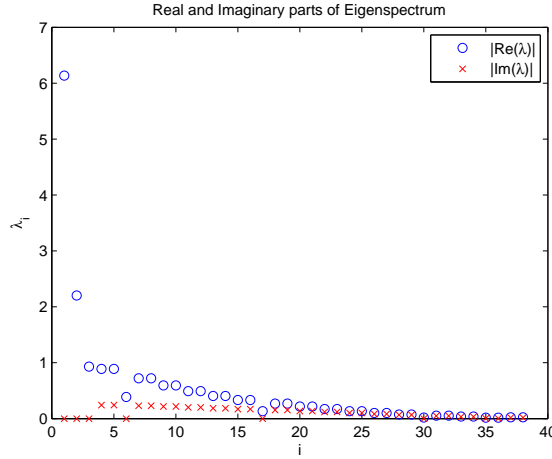
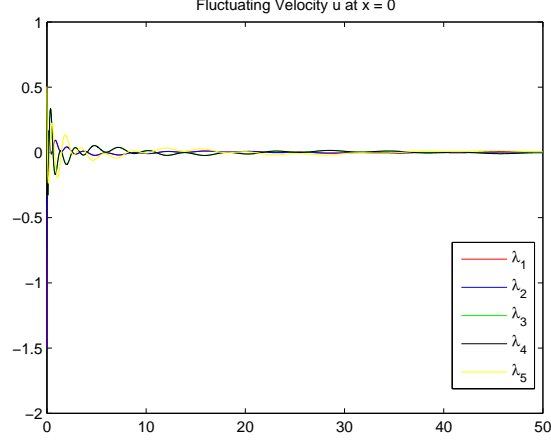


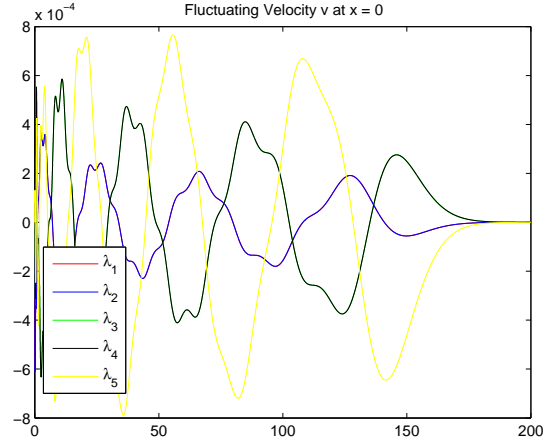
Figure 4:  $|Re(\lambda_i)|$  and  $|Im(\lambda_i)|$  of eigenvalue solutions to (44)

Figure 5 shows plots of some of the  $u'$  and  $v'$  eigenfunction solutions to (44) along the cross section  $x = 0$ .

One can see that, like the series solutions in Figures 1–2, these are *not* non-negative, which in fact they need not be, as they are not root-mean-square (rms) quantities in the governing equation (44). These plots suggest that perhaps the



(a)  $u'$



(b)  $v'$

Figure 5: Laguerre-Galerkin method computed fluctuating velocity eigenfunctions corresponding to the first 5 eigenvalues

approach to take is to compute these eigenfunctions for all the eigenvalues, and then average them. The interpretation would be a velocity fluctuation (*not* an rms velocity), however, unless we average in some way that makes the quantities non-negative.

The procedure/implementation presented in this section has prompted several additional questions that will need to be addressed. First, how do we wish to define the span in (58)? Do we wish to average all the eigenfunctions, or only use a select set of them? Moreover, what do we wish to do with the eigenfunctions that have non-zero imaginary parts? Finally, as in the series solution approach of Section 3.1, there remains the question of how one would go about normalizing the final solution.

## 5 Steady Inhomogeneous Boundary Value Problem (BVP) for the Fluctuations with Mean Flow Residual-Based Source

Given the discussion above of the “pseudotemporal” approach, namely the fact that the approach handles the ill-posedness of the original equations by implicitly introducing a source, making the equations inhomogeneous, the natural question to ask is: is the “pseudotemporally”-generated artificial source that comes from the eigenvalue  $\lambda$  in (43) the “right” source to use? It seems that perhaps one should consider introducing a source directly into the steady fluctuating equations (6).

One idea to generate such a source is using the mean equations (5). Assuming the “standard” boundary simplifications  $(u'u')_x \approx (v'v')_y \approx (u'v')_x \approx VV_y \approx 0$ , and the eddy viscosity model:  $-(u'v')_y = \nu_{eff}U_{yy}$ , as done in the introduction, we obtain Eqns. (7) for the mean quantities. These are closed by the eddy-viscosity model, i.e., they can be solved independently of the fluctuation equations (numerically or possibly analytically).

Since turbulence is inherently a non-linear phenomenon, one may argue that the fluctuation equations need to be non-linear. Let us see if we can come up with a set of “split” fluctuating equations given (2)–(4) and (7). Setting  $(u'u')_x \approx (v'v')_y \approx (u'v')_x \approx VV_y \approx 0$  in (2)–(4), these equations become:

$$\begin{aligned} [U_x + V_y] &+ [u'_x + v'_y] &= 0 \\ [UU_x + VU_y + P_x - \nu_{eff}U_{yy}] &+ [Uu'_x + Vu'_y + u'U_x + v'U_y + p'_x + \nu_{eff}U_{yy} + (u'v')_y] &= 0 \\ [UV_x + VV_y + P_y] &+ [Uv'_x + Vv'_y + u'V_x + v'V_y + p'_y] &= 0 \end{aligned} \quad (60)$$

Now, from (60), the “natural” set of (nonlinear!) set of equations for the fluctuations is:

$$\boxed{\begin{aligned} u'_x + v'_y &= 0 \\ Uu'_x + Vu'_y + u'U_x + v'U_y + p'_x + (u'v')_y &= -\nu_{eff}U_{yy} \\ Uv'_x + Vv'_y + u'V_x + v'V_y + p'_y &= 0 \end{aligned}} \quad (61)$$

Note that (61) is simply an example of the sort of inhomogeneous non-linear fluctuating equations one can come up with assuming a mean-flow-residual-based source. These equations would be different if one did not wish to make the boundary layer simplifications  $(u'u')_x \approx (v'v')_y \approx (u'v')_x \approx VV_y \approx 0$ , for instance. One could also add an unsteady term to (61), if desired.

Several things are noteworthy about the equations (61) and variants of these equations. First, the equations are *non-linear* in the fluctuations, due to the presence of the Reynolds stress term  $(u'v')_y$  in the  $x$ -momentum equation. This is very promising, as turbulence is inherently non-linear, and hence non-linear equations are more likely to capture appropriately the physics of turbulent flow. Moreover, the equations (61) are *inhomogeneous*, and so with homogeneous boundary conditions at  $y = 0, \infty$ , will be well-posed. Thus, there is no reason to formulate the equations as an eigenvalue problem (EVP) for well-posedness. Similarly, if we believe our flow is steady, there is no need to introduce an “artificial” source or “pseudo-time” term for well-posedness, as done in Section 4.

Note that, in the equations (61), we have retained the intuitively appealing property of the “splitting” that [split mean equations] + [split fluctuating equations] = [original equations (2)–(4)]. Given  $U$  and  $V$ , computed by solving (7) with a model for  $\nu_{eff}$ , the equations (61) are closed, and can be solved numerically. Numerical solution of (61) has yet to be implemented. These equations can be discretized using a Laguerre-Galerkin method like the one presented in Section 4.1. Note, however, that a Newton step would need to be added to handle the non-linearity that has been introduced.

## 6 Conclusions, Remaining Issues, Future Considerations

In the present document, we have presented several variants of equations for the fluctuating velocities and pressure, which we are interested in computing for the sake of calculating turbulent pressure loads on a body. At the heart of the equations is the mean/fluctuation “splitting”, first proposed in [4, 5]: following the decomposition of the relevant field

into the sum of a mean and a fluctuation (1) and substitution of this sum into the governing equations (taken at the present time to be incompressible 2D Euler equations), one splits the fluctuating equations (6), normally lost to averaging in RANS, from the mean equations (5) and solves them independently for the desired fluctuations. Given some turbulence model to close the mean equations to yield, e.g. (7), one may solve for the mean quantities independently from the fluctuations, either numerically or perhaps analytically, and feed the computed mean quantities into (5), from which the fluctuations can be obtained.

A crucial observation is that, as they stand, the equations (6) are *ill*-posed: they are homogeneous equations with homogeneous boundary conditions. Our task herein has been to propose and analyze several approaches for making the equations (6) well-posed. This can be done by:

- Assuming an Orr-Sommerfeld-like ansatz to derive a Sturm-Liouville eigenvalue problem for the equations (6) (Section 3).
- Adding an unsteady term to the fluctuating equations, and formulate a “pseudotemporal” eigenvalue problem for (42) (Section 4).
- Adding a source to the homogeneous equations (6), e.g., a source based on some residual coming from the mean equations (Section 5).

Preliminary numerical experiments have revealed the following:

- The solutions to the “artificial” eigenvalue problem (21) with an ad hoc eigenvalue scaling factor  $\lambda$ , like proposed in [4, 5], are difficult to interpret physically. In particular, as the eigenvalue is not tied to a particular solution form or ansatz, it is unclear how to make sense of complex eigenvalues and eigenfunctions. Alternate, more naturally arising EVPs are sought therefore (Sections 3–4).
- Regardless of the final boundary value or eigenvalue problem selected, an amenable spatial discretization is a Laguerre-Galerkin spectral basis (Section 4.1), as the BVP/EVP is posed on a semi-infinite domain  $\Omega = (0, \infty) \times (0, \infty)$ . We have derived in the present work a Galerkin basis comprised of weighted Laguerre polynomials, namely (52), that satisfies the relevant boundary conditions, and can be employed in the ultimate implementation.
- Given that our domain is semi-infinite and the connection to Sturm-Liouville theory/orthogonal polynomials exhibited in Sections 2–4, it seems most appropriate to try to connect the streamfunction EVP to Laguerre’s differential equation

$$(xe^{-x}u')' + ne^{-x}u = 0 \quad (62)$$

posed on a semi-infinite domain  $(0, \infty)$  rather than Fourier’s equation ( $u'' + \lambda u = 0$ ), as was attempted in earlier works [4, 5]. A direct connection between approximate analytical solutions to the streamfunction EVP and Laguerre’s equation (62) is exhibited in [7].

- If one is satisfied with solving the problem on a large finite domain  $(0, L) \times (0, L)$  for  $L \gg 1$ , then a Hermite cubic finite element method, described in detail in [3] can also be employed. Note that finite element shape functions that are continuous and have continuous first derivatives are required to solve EVP/BVPs for the streamfunction (e.g., (44)). In particular, *linear* finite elements will be inadequate. This may have some implications if the EVP is to be implemented in SIERRA, which currently only supports linear finite elements.
- Since the operator governing the EVP will almost surely be asymmetric, there is no guarantee that the eigenvalues and corresponding eigenfunctions that solve the EVP will be real. It will need to be decided, therefore, how to interpret the complex eigenvalue/eigenfunction solutions in a physical context.
- Previously [4, 5], an association was made between the solutions to the fluctuating equations (6) and root-mean-square (rms) quantities, based on the observation that there is no explicit temporal scale in these expressions. Based on this connection, one way of validating the equations was to require that their solutions be non-negative:  $u', v', p' \geq 0$ . However, in fact, the equations (6) were *not* derived for rms quantities; therefore there is no reason for the solutions to (6), or EVP variants of (6) to be necessarily non-negative. One therefore needs an interpretation of what these fluctuations mean exactly, so as to have a means of interpret as well as validate the solutions to (6).

- Given the point above, the eddy viscosity  $\nu_{eff}$  in the mean equations (7) may need to be adjusted, to account for the fact that we are employing the model  $-(u'v')_y = \nu_{eff}U_{yy}$ , where the left-hand-side is *not* the usual Reynolds stress (i.e., it is not averaged).
- Since the equations (6) are homogeneous, any constant multiple of a computed solution to (6) will also solve these equations. Some technique will need to be devised to somehow normalize (i.e., fix the magnitude of) the solutions.
- It may be worthwhile to impose boundary conditions on the fluctuating quantities in the streamwise direction, i.e., at  $x = 0$  and  $x = \infty$ .

Overall, significant progress has been made in identifying issues involving the well-posedness, numerical solvability, and physical correctness of the fluctuating equations (6). We end by enumerating several further extensions that it may be desirable to consider in the future, as the model is augmented to take into account more complicated flow scenarios:

- Adding viscous terms to (5) and (6).
- Considering the compressible equations of fluid mechanics (compressible Euler or Navier-Stokes equations).
- Formulating non-linear variants of (6), such as what is proposed in Section 5. (This seems crucial from a physics perspective, as turbulence is inherently a non-linear phenomenon.)
- Rather than employing the somewhat ad hoc “pseudotemporal” approach (Section 4), making (6) well-posed by adding some physically-relevant source to these equations (e.g., as in Section 5).
- Performing all the analysis and formulating the problem in the primitive variables  $u'$ ,  $v'$  and  $p'$ , rather than working with the streamfunction  $\psi'$ , as this latter approach is limited to two-dimensions (2D).

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